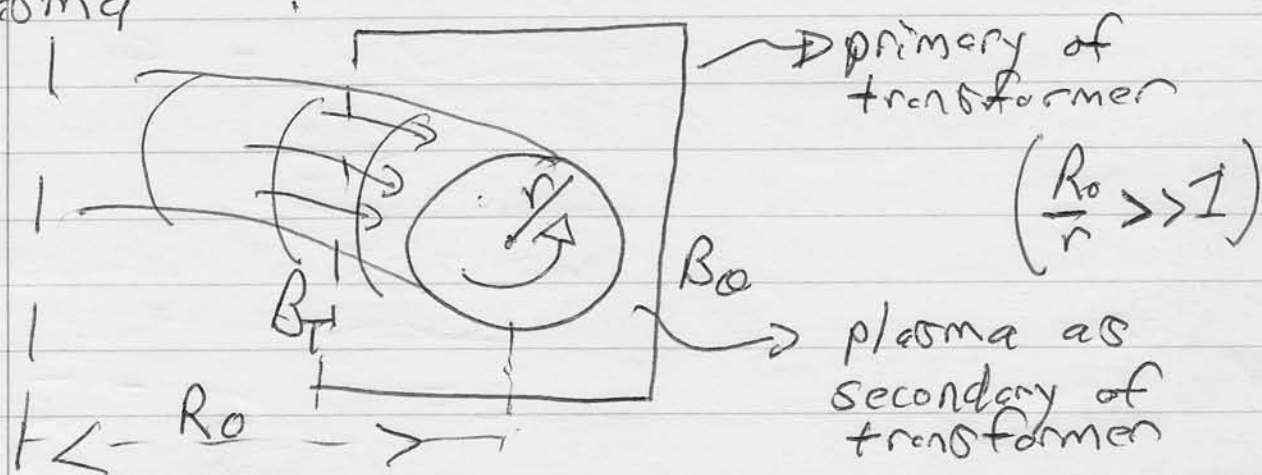


(Case Study)

side: Magnetic Field Lines on a Tokamak:
A Practical Example of Phase Space
Evolution on Tori

→ What is a Tokamak?

- toroidal confinement device for magnetized plasma



$$I = I_0 \Rightarrow B_T(R) = \frac{2I_0}{R} \rightarrow \text{toroidal field (external)}$$

$$B_T \gg B_\theta \quad B_\theta(r) = \int_0^r dr' \frac{J_T(r')}{c} \rightarrow \text{poloidal field (plasma current)}$$

- $B_\theta(r) \rightarrow$ shorts out charge separation due to ∇B drift
- \rightarrow stability, confinement
- \rightarrow heating (ohmic)

more info: "Tokamak Plasma, A Complex Physical System"
B.B. Kadomtsev

Minimal Model: The "Toroidal Cow" for Toroidal Field Configurations

- tokamak as periodic cylinder with;
- $$\left. \begin{array}{l} 0 < r < a \\ L = 2\pi R_0 \end{array} \right\} \begin{array}{l} B_z = B_T \quad (\text{uniform, external}) \\ B_\theta(r) \end{array}$$

$$\langle \underline{B} \rangle = B_\theta(r) \hat{\theta} + B_z \hat{z}$$

$$\underline{B} = \langle \underline{B} \rangle + \underline{\tilde{B}}_\perp$$

- field line: $\frac{dz}{B_T} = \frac{rd\theta}{B_\theta(r) + \tilde{B}_\theta} = \frac{dr}{\tilde{B}_r}$

∴

$$\frac{d\theta}{dz} = \frac{1}{r} \frac{B_\theta(r) + \tilde{B}_\theta}{B_z} \approx \frac{1}{r} \frac{B_\theta(r)}{B_z} \quad \tilde{B}_\theta \ll B_\theta(r)$$

$$\frac{dr}{dz} = \frac{\tilde{B}_r}{B_z}$$

For un-perturbed field configuration:

$$\frac{d\theta}{dz} = \frac{1}{r} \frac{B_\theta(r)}{B_z} = \frac{1}{R q(r)} ; \quad q(r) \equiv B_z r / R B_\theta(r)$$

↓
safety factor

$$\frac{dr}{dz} = 0 \quad (\text{no radial wandering})$$

$\Gamma(r) \equiv$ winding rate (i.e. rotational transform)
(# poloidal circuits per toroidal)

→ Relation to Hamiltonian Dynamics \mathcal{P} .

$$\frac{dx}{dz} = \frac{\tilde{B}_\theta}{B} \quad , \quad \frac{dy}{dz} = \frac{B_y(x) + \tilde{B}}{B} \quad \nabla \cdot \tilde{V} = 0 \Rightarrow \text{Ham.}$$

Useful to observe similarity between:

a) Hamiltonian System with:

$$H = H(x, y) \quad \text{so} \quad \begin{cases} \dot{x} = -\partial H / \partial y \\ \dot{y} = \partial H / \partial x \end{cases} \quad (\nabla \cdot \tilde{V}_T = 0)$$

so Liouville Egn. for $f(t, x, y)$ is:

$$\frac{\partial f}{\partial t} + \dot{x} \frac{\partial f}{\partial x} + \dot{y} \frac{\partial f}{\partial y} = 0$$

$$\therefore \frac{\partial f}{\partial t} - \frac{\partial H}{\partial y} \frac{\partial f}{\partial x} + \frac{\partial H}{\partial x} \frac{\partial f}{\partial y} = 0$$

Can further specialize: $H = H_0(x) + \tilde{H}(x, y)$

$$\Rightarrow \begin{cases} \dot{x} = -\partial \tilde{H} / \partial y \\ \dot{y} = \frac{\partial H_0}{\partial x} + \frac{\partial \tilde{H}}{\partial x} \end{cases}$$

and

$$\frac{\partial f}{\partial t} + \frac{\partial H_0}{\partial x} \frac{\partial f}{\partial y} - \frac{\partial \tilde{H}}{\partial y} \frac{\partial f}{\partial x} + \frac{\partial \tilde{H}}{\partial x} \frac{\partial f}{\partial y} = 0$$

$$\Rightarrow \frac{\partial f}{\partial t} + v_y(x) \frac{\partial f}{\partial y} + \{ \tilde{H}, f \} = 0$$

eg. $\begin{cases} H = \phi(r, \theta) \\ \text{G.C. plasma} \end{cases}$

b) Equation for Magnetic Flux $\psi(r, \theta)$

$$\underline{B} = B_0 \underline{\hat{z}} + \underline{\nabla} \psi \times \underline{\hat{z}} \rightarrow \underline{B} \text{ field} \quad \psi = A_z(r, \theta)$$

$$\psi = \langle \psi(r) \rangle + \tilde{\psi}(r, \theta) \rightarrow \text{Magnetic Flux function}$$

then, by definition:

$$\underline{B} \cdot \underline{\nabla} \psi = 0$$

(Flux constant along magnetic field lines)

\Rightarrow

$$\left(B_0 \frac{\partial}{\partial z} + \frac{B_0(r)}{r} \frac{\partial}{\partial \theta} + \underline{\tilde{B}}_1 \cdot \underline{\nabla}_1 \right) \psi = 0$$

\approx

$$\left(\frac{\partial}{\partial z} + \frac{1}{R_0(r)} \frac{\partial}{\partial \theta} + \frac{\underline{\tilde{B}}_1 \cdot \underline{\nabla}_1}{B_0} \right) \psi = 0$$

can read off analogy: (isomorphism)

$$\{ z \leftrightarrow t, \quad r \leftrightarrow x, \quad r d\theta \leftrightarrow y \}$$

$$\left\{ \begin{array}{l} 1/Rz(r) \leftrightarrow v_y(x) \leftrightarrow \omega(I) \end{array} \right.$$

$$\left\{ \begin{array}{l} \frac{\partial \omega(I)}{\partial I} \neq 0 \Rightarrow z'(r) \neq 0 \text{ "shear"} \\ \text{(winding rate varies with radius)} \end{array} \right.$$

$$\{ \langle v_y(x) \rangle \leftrightarrow B_\theta(r) \}$$

$$\{ \tilde{B}_I \leftrightarrow \nabla \tilde{H} \times \tilde{z} \}$$

$$\left\{ \begin{array}{l} \nabla_{\perp} \cdot \tilde{B}_I = 0 \\ \nabla \cdot (\nabla \tilde{H} \times \tilde{z}) = 0 \end{array} \right.$$

Liouville Thm.

($\nabla \cdot \tilde{B} = 0$ underlies Hamiltonian structure)

Thus, Hamiltonian trajectory on 2-torus in phase space (for 2 degs. freedom) equivalent to trajectory of magnetic field line on torus of radius (minor) = r in space!

$$\left\{ \begin{array}{l} r \leftrightarrow I \\ 1/Rz(r) \leftrightarrow \omega(I) \\ \theta \leftrightarrow \theta \quad \text{(angle variable)} \end{array} \right.$$

An Observation

Consider solution of flux equation perturbatively
i.e.

$$\underline{B} = \underline{B}_0 + \underline{\tilde{B}} \quad \underline{B}_0 = B_0 \underline{\hat{z}} + B_0 \underline{\hat{\theta}}$$

$$\psi = \langle \psi(r) \rangle + \tilde{\psi}$$

$$\underline{B} \cdot \nabla \psi = 0 \Rightarrow$$

$$(\underline{B}_0 \cdot \nabla) \tilde{\psi} = -\tilde{B}_r \frac{\partial \langle \psi(r) \rangle}{\partial r}$$

expand $\tilde{\psi}$, \tilde{B}_r as:

$$\tilde{B}_r = \sum_{m,n} \tilde{B}_r(r)_m e^{i(m\theta - n\phi)}$$

$$z \rightarrow R\phi$$

$$\Rightarrow \left(-\frac{in}{R} B_0 + \frac{im}{r} B_0 \right) \tilde{\psi}_n = -\tilde{B}_{r,n} \frac{\partial \langle \psi(r) \rangle}{\partial r}$$

$$\tilde{\psi}_{m,n}(r) = \frac{-\tilde{B}_{r,n}(r) \frac{\partial \langle \psi(r) \rangle}{\partial r}}{-i \frac{B_0}{R} \left(n - \frac{m}{\Sigma(r)} \right)}$$

0. $\tilde{\Psi}_{m,n}(r) = i R \left(\frac{\tilde{B}_{nm}(r)}{B_0} \right) \partial \langle \Psi(r) \rangle / \partial r$

$\left(n - \frac{m}{q(r)} \right) \rightarrow$ small divisor problem \int

\Rightarrow perturbative solution diverges at $q(r) = m/n \Rightarrow$ defines resonant surface (special tori)

c.e. $\left\{ \begin{array}{l} \text{radius where pitch of field line } (q(r)) \\ \text{resonates with pitch of perturbation} \\ (m/n) \end{array} \right.$

\Rightarrow linear solution to Liouville Egn. fails here \downarrow
 $\underline{n \cdot \omega} = 0$
 (resonances)

Welcome to small divisor problem \int